

# Recovering Non-negative and Combined Sparse Representations

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## Abstract

The non-negative solution to an underdetermined linear system can be uniquely recovered sometimes, even without imposing any additional sparsity constraints. In this paper, we derive conditions under which a unique non-negative solution for such a system can exist, based on the theory of polytopes. Furthermore, we develop the paradigm of combined sparse representations, where only a part of the coefficient vector is constrained to be non-negative, and the rest is unconstrained (general). We analyze the recovery of the unique, sparsest solution, for combined representations, under three different cases of coefficient support knowledge: (a) the non-zero supports of non-negative and general coefficients are known, (b) the non-zero support of general coefficients alone is known, and (c) both the non-zero supports are unknown. For case (c), we propose the combined orthogonal matching pursuit algorithm for coefficient recovery and derive the deterministic sparsity threshold under which recovery of the unique, sparsest coefficient vector is possible. Simulation results for recovery of combined representations using randomly generated dictionaries and coefficients are presented. We show that the basis pursuit algorithm, with partial non-negative constraints, and the proposed greedy algorithm perform better in recovering the unique sparse representation when compared to their unconstrained counterparts.

*Keywords:* underdetermined linear system, sparse representations, non-negative constraints, orthogonal matching pursuit, unique sparse solution

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## 1. Introduction

We investigate the problem of recovering non-negative and combined sparse representations from underdetermined linear models. The system of linear equations with the

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constraint that the solution is non-negative can be expressed as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha}, \text{ where } \boldsymbol{\alpha} \geq 0, \quad (1)$$

where  $\mathbf{y} \in \mathbb{R}^M$  is the data vector,  $\boldsymbol{\alpha} \in \mathbb{R}^{K_x}$  is the non-negative solution (coefficient vector), and  $\mathbf{X} \in \mathbb{R}^{M \times K_x}$  is the dictionary with  $K_x > M$ . When only a part of the solution is constrained to be non-negative and the rest is unconstrained (general), we obtain the combined representation model,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{D}\boldsymbol{\beta}, \text{ where } \boldsymbol{\alpha} \geq 0. \quad (2)$$

Here, the coefficient vector  $\boldsymbol{\beta} \in \mathbb{R}^{K_d}$  is unconstrained, and  $\mathbf{X} \in \mathbb{R}^{M \times K_x}$  and  $\mathbf{D} \in \mathbb{R}^{M \times K_d}$  are the sub-dictionaries for the non-negative and general representations respectively. We denote the combined coefficient vector as  $\boldsymbol{\delta} = [\boldsymbol{\alpha}^T \boldsymbol{\beta}^T]^T$ , and the combined dictionary as  $\mathbf{G} = [\mathbf{X} \ \mathbf{D}]$ . We assume that  $\mathbf{G}$  is overcomplete with  $K_x + K_d > M$ , and the columns of the dictionaries are normalized to have unit  $\ell_2$  norm. The sparsest solutions to (1) and (2) are obtained by minimizing the  $\ell_0$  norm, the number of non-zero elements, of the corresponding non-negative coefficient vector,  $\boldsymbol{\alpha}$ , or the combined coefficient vector,  $\boldsymbol{\delta}$ . In both the cases, the unique minimum  $\ell_0$  norm solution, when it exists, will be referred to as  $ML_0$  solution. In this paper, we focus on obtaining deterministic guarantees for recovery of the  $ML_0$  solutions to the linear systems (1) and (2), using both convex and greedy algorithms, based on the properties of the dictionaries.

Some of the applications of the non-negative representation model in (1), and the combined model in (2) are in image inpainting [1], automatic speech recognition using exemplars [2], protein mass spectrometry [3], astronomical imaging [4], spectroscopy [5], source separation [6], and clustering/semi-supervised learning of data [7, 8], to name a few.

### 1.1. Prior Work

For the non-negative representation model in (1), a sufficiently sparse  $ML_0$  solution can be recovered by minimizing the  $\ell_1$  norm of  $\boldsymbol{\alpha}$ , using the non-negative version of the basis pursuit (BP) algorithm [9], which we refer to as NN-BP. The optimization program can be expressed as

$$\min_{\boldsymbol{\alpha}} \mathbf{1}^T \boldsymbol{\alpha} \text{ subject to } \mathbf{y} = \mathbf{X}\boldsymbol{\alpha}, \boldsymbol{\alpha} \geq 0. \quad (3)$$

The conditions on  $\mathbf{X}$  under which the recovery of  $ML_0$  solution using (3) is possible have been derived based on the neighborliness of polytopes [10, 11, 12], and the non-negative null-space property [13]. A non-negative version of the greedy orthogonal matching pursuit (OMP) algorithm [14], which we will refer to as NN-OMP, for recovering the coefficients has also been proposed [15]. If the set

$$\{\boldsymbol{\alpha} | \mathbf{y} = \mathbf{X}\boldsymbol{\alpha}, \boldsymbol{\alpha} \geq 0\} \quad (4)$$

contains only one solution, we can use any variational function instead of the  $\ell_1$  norm in order to obtain the unique non-negative solution [11, 12, 15]. In particular, the solution can be obtained by using the non-negative least squares (NNLS) algorithm [3, 16].

A major part of our work investigates the combined sparse representation model introduced in (2), where only a part of the sparse coefficient vector is constrained to be non-negative. We consider the *deterministic sparsity thresholds* i.e., the maximum number of non-zero coefficients possible in the  $ML_0$  solution, such that the  $ML_0$  solution can be uniquely recovered. To the best of our knowledge, such an investigation has not been reported so far in the literature. However, when both  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are unconstrained general sparse vectors, the sparsity thresholds for recovery of the  $ML_0$  solution have been presented in [17, 18]. By considering the coherence parameters of  $\mathbf{X}$  and  $\mathbf{D}$  separately, the authors in [17] show that an improvement up to a factor of two can be achieved in the deterministic sparsity threshold when compared to considering  $\mathbf{X}$  and  $\mathbf{D}$  together as a single dictionary. Note that deterministic sparsity thresholds provide guarantees that hold for all sparsity patterns and non-zero values in the coefficient vectors. *Probabilistic* or *robust* sparsity thresholds, that hold for most sparsity patterns and non-zero values in the coefficient vectors have also been derived in [17], again for the case where  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are general sparse vectors. When this representation is approximately sparse and corrupted by additive noise, theory and algorithms for coefficient recovery are presented in [19].

## 1.2. Contributions

We present deterministic recovery guarantees for both the non-negative and the combined sparse representation models given by (1) and (2) respectively. Furthermore, we propose a greedy algorithm for performing coefficient recovery in combined representations and derive deterministic sparsity thresholds for unique recovery using  $\ell_1$  minimization and the proposed greedy algorithm.

For the non-negative model in (1), we derive the sufficient conditions for (4) to be singleton based on the neighborliness properties of the quotient polytope corresponding to the dictionary  $\mathbf{X}$ . Similar analyses reported in [11, 12] assume that the dictionary  $\mathbf{X}$  is obtained from a random ensemble and append a row of ones to it, such that the row span of  $\mathbf{X}$  contains the vector  $\mathbf{1}^T$ . In contrast, we do not assume any randomness on  $\mathbf{X}$  and only require that its row span intersects the positive orthant. We show that the sparsity threshold on  $\boldsymbol{\alpha}$ , for the set (4) to be singleton, is the same as the deterministic sparsity threshold for recovering the  $ML_0$  solution of a general sparse representation. Whenever this threshold is satisfied,  $\ell_1$ -norm regularization in (3) can be replaced with any variational function. Section 2 presents the analysis of the non-negative representation model.

For the combined model in (2), we propose a variant of the greedy OMP algorithm, the combined OMP (COMB-OMP) algorithm, for performing coefficient recovery. We also consider a  $\ell_1$  regularized convex algorithm, which we refer to as combined BP (COMB-BP). We derive the deterministic sparsity thresholds for recovering the  $ML_0$  solution using both the COMB-BP and COMB-OMP algorithms. We show that a factor-of-two improvement in the sparsity threshold, observed when  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  are general sparse vectors [17], holds for recovery using the COMB-BP also. We also show that such an improvement in the sparsity threshold cannot be observed using the COMB-OMP algorithm, because of the partial non-negativity constraint on the coefficient vector. However, COMB-OMP incurs very low computational complexity when compared to COMB-BP. Furthermore, we obtain the sparsity thresholds in the following cases of coefficient support knowledge: (a) the non-zero support of both  $\boldsymbol{\alpha}$ ,  $\boldsymbol{\beta}$  are known, and (b) non-zero support of  $\boldsymbol{\beta}$  alone is known. When analyzing case (b), we factor out the contribution of the general representation component and arrive at conditions under which  $\ell_1$ -norm regularization in the resulting optimization can be replaced with any variational function for the recovery of  $\boldsymbol{\alpha}$ . Section 3 presents all the details in the analysis of the combined representation model.

The performance of the COMB-BP and the COMB-OMP algorithms are also analyzed using simulations. The dictionary  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are obtained either from uniform distribution or fixed as random signs ( $\pm 1$ ). It is shown that both COMB-BP and COMB-OMP respectively perform better than their unconstrained counterparts, the BP and the OMP, particularly as the  $K_x$  be-

comes larger. We also show that the COMB-OMP incurs substantially less computational complexity when compared to the COMB-BP.

### 1.3. Notation

Lowercase boldface letters denote column vectors and uppercase boldface denote matrices, e.g.,  $\mathbf{a}$  and  $\mathbf{A}$  denote a vector and a matrix respectively.  $\mathbf{a}_i$  indicates the  $i^{th}$  column of the matrix  $\mathbf{A}$ . The Moore-Penrose pseudoinverse of a matrix  $\mathbf{A}$ , given by  $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$  is denoted as  $\mathbf{A}^\dagger$ . The notation  $\mathbf{A} = \text{diag}(\mathbf{a})$  means that  $\mathbf{A}$  is a diagonal matrix with the elements of the vector  $\mathbf{a}$  as its diagonal.  $|\mathbf{A}|$  refers to a matrix whose elements are the absolute values of the elements of  $\mathbf{A}$  and the same notation applies to vectors also. The maximum row and maximum column sums of  $\mathbf{A}$  are referred to as  $\|\mathbf{A}\|_{\infty, \infty}$  and  $\|\mathbf{A}\|_{1, 1}$  respectively. A set is denoted as  $\mathcal{A}$ , its cardinality is given by  $|\mathcal{A}|$  and its complement by  $\mathcal{A}^c$ . The operator  $[\cdot]^+$  returns the maximum of the argument and zero, and  $\max(\cdot, \cdot)$  returns the maximum of the two arguments.  $\mathbf{I}_K$  denotes an identity matrix of size  $K \times K$ ,  $\mathbf{1}_{K_1, K_2}$  is a matrix of ones with size  $K_1 \times K_2$ . Similar notation will be employed for defining vectors also. When it is clear from context, the subscripts will be dropped for simplicity.

## 2. Non-negative Sparse Representations

For the non-negative representation given in (1), we denote the number of non-zero coefficients in  $\boldsymbol{\alpha}$  as  $S_x$ .

**Definition 1.** ([20]) The two-sided coherence (or simply coherence) of the dictionary  $\mathbf{X}$  is

$$\mu_x = \max_{i \neq j} \frac{|\mathbf{x}_i^T \mathbf{x}_j|}{\|\mathbf{x}_i\|_2 \|\mathbf{x}_j\|_2}, \quad (5)$$

**Definition 2.** ([15]) The one-sided coherence of the dictionary  $\mathbf{X}$  is

$$\sigma_x = \max_{i \neq j} \frac{|\mathbf{x}_i^T \mathbf{x}_j|}{\|\mathbf{x}_i\|_2^2}. \quad (6)$$

If the columns of  $\mathbf{X}$  are normalized, we have  $\mu_x = \sigma_x$ , and if they had different  $\ell_2$  norms, we would have  $\mu_x \leq \sigma_x$  [15, Lemma 1].

**Definition 3.** ([15]) The dictionary  $\mathbf{X}$  belongs to the class of matrices denoted as  $\mathcal{M}^+$ , if its row span intersects the positive orthant.

If  $\mathbf{X} \in \mathcal{M}^+$ ,  $\exists \mathbf{h}$  such that  $\mathbf{h}^T \mathbf{X} = \mathbf{w}^T$ ,  $\mathbf{w} > \mathbf{0}$ . Let us define  $\mathbf{W} = \text{diag}(\mathbf{w})$ ,  $\mathbf{U} = \mathbf{XW}^{-1}$ , and denote  $\sigma_u$  and  $\mu_u$  as the one-sided and two-sided coherences of  $\mathbf{U}$  respectively. In [12, Theorem 1] it is shown that  $\mathbf{X} \in \mathcal{M}^+$  is a necessary condition for (4) to be singleton. The main result in [15, Theorem 2] states that the set (4) will be singleton if  $\mathbf{X} \in \mathcal{M}^+$  and  $S_x < 0.5(1 + 1/\sigma_u)$ .

We will now state the main result of this section, whose proof will be relegated to the end of this section.

**Theorem 2.1.** *When  $\mathbf{X} \in \mathcal{M}^+$ , the set defined in (4) is singleton if the number of non-zero entries in  $\boldsymbol{\alpha}$ ,  $S_x < 0.5(1 + 1/\mu_x)$ .*

The threshold given in the above theorem is better than that of [15, Theorem 2], because  $\mu_x = \mu_u$ ,  $\mu_u \leq \sigma_u$  and hence  $\mu_x \leq \sigma_u$ . We are able to improve the threshold by resorting to geometric arguments based on the theory of polytopes. The rest of this section will state and prove lemmas that will be used in the proof of our main result.

We will define three geometric entities, the cross-polytope  $\mathcal{C}^{K_x}$ , the simplex  $\mathcal{T}^{K_x-1}$  and the positive orthant  $\mathbb{R}_+^{K_x}$ , that will be used in the proof. The cross-polytope is defined as the  $\ell_1$  ball,  $\|\boldsymbol{\alpha}\|_1 \leq 1$ , in  $\mathbb{R}^{K_x}$ , and  $\mathcal{T}^{K_x-1}$  is the standard simplex, the convex hull of unit basis vectors. Any general sparse representation with  $S_x$  non-zero coefficients can be successfully recovered using  $\ell_1$  minimization (BP), if the quotient polytope  $\mathbf{XC}^{K_x}$  is *centrally  $S_x$ -neighborly* [21, Theorem 1]. This form of neighborliness implies that any set of  $S_x$  vertices of  $\mathbf{XC}^{K_x}$ , not including an antipodal pair (pair of  $\pm \mathbf{x}_i$ ), span a face. For unique recovery of non-negative  $S_x$ -sparse vectors using the linear program given in (3), the condition on the quotient polytope  $\mathbf{XT}$  is that it must be *outwardly  $S_x$ -neighborly* [10, Theorem 1]. Here, we fix  $\mathcal{T} = \mathcal{T}^{K_x-1}$  if  $\mathbf{0}$  can be expressed as a convex combination of the columns of  $\mathbf{X}$ , else we fix  $\mathcal{T} = \mathcal{T}_0^{K_x}$  where  $\mathcal{T}_0^{K_x}$  is the solid simplex, the convex hull of  $\mathcal{T}^{K_x-1}$  and the origin. When every set of  $S_x$  vertices, not including the origin, span a face, the quotient polytope is said to be outwardly  $S_x$ -neighborly.

**Lemma 2.2.** *When  $\mathbf{X} \in \mathcal{M}^+$  and the number of non-zero coefficients in  $\boldsymbol{\alpha}$  is  $S_x$ , the set defined in (4) is singleton if the quotient polytope  $\mathbf{XT}_0^{K_x}$  is outwardly  $S_x$ -neighborly.*

PROOF. By assumption,  $\exists \mathbf{h}$  such that  $\mathbf{h}^T \mathbf{X} = \mathbf{w}^T$ ,  $\mathbf{w} > \mathbf{0}$ . Consider the quotient polytope  $\mathbf{UT}_0^{K_x}$ , where  $\mathbf{U} = \mathbf{XW}^{-1}$  and  $\mathbf{W} = \text{diag}(\mathbf{w})$ . Since  $\mathbf{XT}_0^{K_x}$  is outwardly

$S_x$ -neighborly and the positive scaling of vertices does not affect the neighborliness of a polytope,  $\mathbf{U}\mathcal{T}_0^{K_x}$  is also outwardly  $S_x$ -neighborly. Now denote  $\hat{\mathbf{y}} = \mathbf{y}/(\mathbf{h}^T \mathbf{y})$  and  $\gamma = \mathbf{W}\alpha/(\mathbf{h}^T \hat{\mathbf{y}})$ . The set defined in (4) has a one-to-one correspondence with

$$\{\gamma | \hat{\mathbf{y}} = \mathbf{U}\gamma, \gamma \geq 0\}, \quad (7)$$

If we show that (7) is singleton, then (4) is singleton as well.

Since we know that  $\|\alpha\|_0 = S_x$ , this implies that  $\|\gamma\|_0 = S_x$ . Because of the neighborliness of the quotient polytope  $\mathbf{U}\mathcal{T}_0^{K_x}$ ,  $\hat{\mathbf{y}}$  lies in its simplicial face  $F$  of affine dimension  $S_x$ . Denote  $\mathcal{V}$  to be the set of vertices of  $F$ . The remaining vertices in the quotient polytope are denoted by the set  $\mathcal{V}^c$ . Consider an arbitrary vector  $\hat{\mathbf{y}}^c$  expressed as a convex combination of the vertices  $\mathcal{V}^c$ . Since  $F$  is a face, there exists a linear functional  $\lambda_F$  and a constant  $c$  such that  $\lambda_F^T \hat{\mathbf{y}} = c$  and  $\lambda_F^T \hat{\mathbf{y}}^c < c$  [21]. This means that for an arbitrarily chosen  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{y}}^c$  which are convex combinations of vertices  $\mathcal{V}$  and  $\mathcal{V}^c$  respectively,  $\|\hat{\mathbf{y}} - \hat{\mathbf{y}}^c\|_2 > 0$ . By extension, the rays in the directions of  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{y}}^c$  intersect only at the origin. The convex cones formed by the vertices  $\mathcal{V}$  and  $\mathcal{V}^c$  are denoted as  $\mathbf{U}_{\mathcal{V}}\mathbb{R}_+^{|\mathcal{V}|}$  and  $\mathbf{U}_{\mathcal{V}^c}\mathbb{R}_+^{|\mathcal{V}^c|}$  respectively. Since  $\|\hat{\mathbf{y}} - \hat{\mathbf{y}}^c\|_2 > 0$  is true for arbitrary pairs of  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{y}}^c$ , the relative interiors of  $\mathbf{U}_{\mathcal{V}}\mathbb{R}_+^{|\mathcal{V}|}$  and  $\mathbf{U}_{\mathcal{V}^c}\mathbb{R}_+^{|\mathcal{V}^c|}$  are disjoint. Therefore, from [22, Theorem 1.32], there exists a hyperplane passing through the origin that separates the cones properly. From [3, Prop. 1], the existence of such a hyperplane is sufficient for (7), and by extension (4), to be singleton.

### 2.1. Proof of Theorem 2.1

If  $S_x < 0.5(1 + 1/\mu_x)$ , the quotient polytope  $\mathbf{X}\mathcal{C}^{K_x}$  is centrally  $S_x$ -neighborly [21, Corollary 1.1]. Since  $\text{vertices}(\mathcal{T}_0^{K_x}) - \{\mathbf{0}\} \subset \text{vertices}(\mathcal{C}^{K_x})$ , central  $S_x$ -neighborliness of  $\mathbf{X}\mathcal{C}^{K_x}$  implies outward  $S_x$ -neighborliness of  $\mathbf{X}\mathcal{T}_0^{K_x}$ . Note that the vertex  $\mathbf{0}$  will be neglected when considering the outward neighborliness. Combining the assumption that  $\mathbf{X} \in \mathcal{M}^+$ , from Lemma 2.2, the set defined in (4) is singleton.

## 3. Combined Sparse Representations

We now turn to investigate the problem of combined sparse representations, where a part of the coefficient support is constrained to be non-negative. For the combined representation model given in (2), the number of non-zero coefficients and the coefficient

support for  $\boldsymbol{\alpha}$  are given by  $S_x$  and  $\mathcal{X}$  respectively. For  $\boldsymbol{\beta}$ , they are respectively denoted as  $S_d$  and  $\mathcal{D}$ . Let us define the combined representation vector  $\boldsymbol{\delta} = [\boldsymbol{\alpha}^T \boldsymbol{\beta}^T]^T$  and the combined dictionary  $\mathbf{G} = [\mathbf{X} \ \mathbf{D}]$ . The set  $\mathcal{G}$  indexes the non-zero coefficients in  $\boldsymbol{\delta}$ . The length of  $\boldsymbol{\delta}$  is denoted by  $K_g$  and its number of non-zero coefficients is referred to as  $S_g$ . We will refer to the coefficient vector  $\boldsymbol{\delta}$  as the combined representation, since it contains both non-negative and general entries from the coefficient vectors  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  respectively. We will define the cross-coherence between the matrices  $\mathbf{X}$  and  $\mathbf{D}$  as

$$\mu_g = \max_{i,j} \frac{|\mathbf{x}_i^T \mathbf{d}_j|}{\|\mathbf{x}_i\|_2 \|\mathbf{d}_j\|_2}. \quad (8)$$

We will present deterministic sparsity thresholds for recovery of the  $ML_0$  solution of (2) when the coefficient supports are unknown as well as partially known.

### 3.1. Non-Zero Supports of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ Known

The vectors  $\boldsymbol{\alpha}_1 \in \mathbb{R}^{S_x}$  and  $\boldsymbol{\beta}_1 \in \mathbb{R}^{S_d}$  contain the non-zero coefficients of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  indexed by  $\mathcal{X}$  and  $\mathcal{D}$  respectively. The matrices  $\mathbf{X}_1$  and  $\mathbf{D}_1$  contain the columns of  $\mathbf{X}$  and  $\mathbf{D}$  indexed by the sets  $\mathcal{X}$  and  $\mathcal{D}$  respectively. Since the coefficient supports are known, we can express (2) as

$$\mathbf{y} = \mathbf{X}_1 \boldsymbol{\alpha}_1 + \mathbf{D}_1 \boldsymbol{\beta}_1, \quad (9)$$

where  $\boldsymbol{\alpha}_1 \geq 0$ . We define  $\boldsymbol{\delta}_1 = [\boldsymbol{\alpha}_1^T \ \boldsymbol{\beta}_1^T]^T$  and the matrix  $\mathbf{G}_1 = [\mathbf{X}_1 \ \mathbf{D}_1]$ . Recovery can be performed using least squares with inequality constraints (LSI) [23, Chap. 23] as

$$\min_{\boldsymbol{\delta}_1} \|\mathbf{y} - \mathbf{G}_1 \boldsymbol{\delta}_1\|_2 \text{ subject to } \mathbf{I}_{\mathcal{X}} \boldsymbol{\delta}_1 \geq \mathbf{0}, \quad (10)$$

where  $\mathbf{I}_{\mathcal{X}} = [\mathbf{I}_{S_x} \ \mathbf{0}_{S_x, S_g}]$  is the indicator matrix such the constraints  $\mathbf{I}_{\mathcal{X}} \boldsymbol{\delta}_1 \geq \mathbf{0}$  and  $\boldsymbol{\alpha}_1 \geq \mathbf{0}$  are equivalent.

If the matrix  $\mathbf{G}_1$  has full column rank,  $\boldsymbol{\delta}_1$  can be estimated by just using least squares (LS) instead of LSI, as the additional constraint in (10) will not impact the solution. The following theorem presents a sufficient condition for  $\mathbf{G}_1$  to be of full column rank.

**Theorem 3.1.** ([18, 17]) *For the system defined in (9), the matrix  $\mathbf{G}_1 = [\mathbf{X}_1 \ \mathbf{D}_1]$  has full column rank if*

$$S_x S_d < \frac{[1 - \mu_x(S_x - 1)]^+ [1 - \mu_d(S_d - 1)]^+}{\mu_g^2}. \quad (11)$$



### 3.2. Non-Zero Support of $\beta$ Alone Known

We will now consider the case where the non-zero support of  $\beta$  given by the set  $\mathcal{D}$  is known for the system in (2). We will derive conditions for unique recovery of  $\alpha$  using NN-BP and NNLS. With the knowledge of non-zero support of  $\beta$ , we can rewrite (2) as

$$\mathbf{y} = \mathbf{X}\alpha + \mathbf{D}_1\beta_1. \quad (12)$$

Define  $\mathbf{P}_{\mathcal{D}}$  to be the projection matrix for the subspace orthogonal to the column space of  $\mathbf{D}_1$ , i.e.,

$$\mathbf{P}_{\mathcal{D}} = \mathbf{I}_M - \mathbf{D}_1\mathbf{D}_1^\dagger. \quad (13)$$

Premultiplying (12) with  $\mathbf{P}_{\mathcal{D}}$ , we get

$$\mathbf{P}_{\mathcal{D}}\mathbf{y} = \mathbf{P}_{\mathcal{D}}\mathbf{X}\alpha \text{ where } \alpha \geq 0. \quad (14)$$

Let us define  $\tilde{\mathbf{y}} = \mathbf{P}_{\mathcal{D}}\mathbf{y}$  and  $\tilde{\mathbf{X}} = \mathbf{P}_{\mathcal{D}}\mathbf{X}$ , such that (14) becomes

$$\tilde{\mathbf{y}} = \tilde{\mathbf{X}}\alpha \text{ where } \alpha \geq 0. \quad (15)$$

The condition for recovery of the unique solution  $\alpha$  from (15) using NN-BP is

$$S_x < 0.5 \left( 1 + \frac{1}{\mu_{\tilde{x}}} \right), \quad (16)$$

where  $\mu_{\tilde{x}}$  is the coherence of  $\tilde{\mathbf{X}}$ .

**Lemma 3.2.** ([18]) *The coherence of  $\tilde{\mathbf{X}}$ , given by  $\mu_{\tilde{x}}$  can be upper bounded as*

$$\mu_{\tilde{x}} \leq 0.5 \left( \frac{[1 - \mu_d(S_d - 1)]^+(1 + \mu_x)}{\mu_x[1 - \mu_d(S_d - 1)]^+ + S_b\mu_g^2} \right). \quad (17)$$

The above lemma follows directly from [18, Theorem 5]. This also implies that for the existence of  $\mathbf{D}_1^\dagger$ , we need to have  $S_d < 1 + 1/\mu_d$ .

**Lemma 3.3.** *Let*

$$\{\hat{\alpha} | \mathbf{y} = \mathbf{X}\hat{\alpha}, \hat{\alpha} \geq 0\} = \{\alpha\}. \quad (18)$$

*For a given non-zero support set  $\mathcal{D}$  of  $\beta$*

$$\{\hat{\alpha} | \tilde{\mathbf{y}} = \tilde{\mathbf{X}}\hat{\alpha}, \hat{\alpha} \geq 0\} = \{\alpha\} \quad (19)$$

*holds if (16) is satisfied, and*

$$\exists \mathbf{h} \text{ such that } \mathbf{h}^T\mathbf{X} > 0 \text{ and } \mathbf{h}^T\mathbf{D}_1 = \mathbf{0}. \quad (20)$$

PROOF. From Theorem 2.1, we know that the singleton condition (19) holds true if (a) the condition in (16) is satisfied, and (b)  $\exists \mathbf{r}$  such that  $\mathbf{r}^T \tilde{\mathbf{X}} > 0$ . Since (18) is true by assumption,  $\exists \mathbf{h}$  such that  $\mathbf{h}^T \mathbf{X} > 0$ . For  $\mathbf{h}^T \mathbf{X} > 0$  and  $\mathbf{r}^T \tilde{\mathbf{X}} > 0$  to hold together, we should have  $\mathbf{h} = \mathbf{P}_{\mathcal{D}}^T \mathbf{r}$ . Therefore, we have  $\mathbf{h}^T \mathbf{D}_1 = \mathbf{0}$ , following the definition of  $\mathbf{P}_{\mathcal{D}}$  in (13).

If the sufficient conditions in Lemma 3.3 are satisfied, NNLS can be used to recover the unique solution of (14), for a given non-zero support  $\mathcal{D}$  of  $\boldsymbol{\beta}$ .

### 3.3. Non-zero Supports of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Unknown

When the supports of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  in (2) are unknown, we will first consider the problem of recovering the coefficients using the convex program,

$$\min_{\boldsymbol{\delta}} \|\boldsymbol{\delta}\|_1 \text{ subject to } \mathbf{y} = \mathbf{G}\boldsymbol{\delta}, \mathbf{I}_{\tilde{\mathcal{X}}}\boldsymbol{\delta} \geq \mathbf{0}, \quad (21)$$

which we refer to as COMB-BP. Here,  $\mathbf{I}_{\tilde{\mathcal{X}}} = [\mathbf{I}_{K_x} \quad \mathbf{0}_{K_x, K_g}]$  is an indicator matrix that picks out  $\boldsymbol{\alpha}$  from the vector  $\boldsymbol{\delta}$  such that the constraint in (21) is equivalent to  $\boldsymbol{\alpha} \geq \mathbf{0}$ . When deriving the threshold on  $S_g$  for the recovery of the  $ML_0$  solution, without loss of generality, we assume that  $S_x \leq S_d$  and  $\mu_x \leq \mu_d$ . Similar thresholds can be derived for the other cases also.

#### 3.3.1. Condition for Recovering the $ML_0$ Solution using the Convex Program

The sufficient condition for the COMB-BP to recover the  $ML_0$  solution is

$$\max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^\dagger \mathbf{g}_i\|_1 < 1. \quad (22)$$

This condition is same as the one given in [24, Theorem 3.3] for recovery of a general sparse vector using BP, since the  $\ell_1$  norm does not depend on the sign of the coefficients. From [17, Theorem 3], the condition (22) can be expressed as

$$(1 + \mu_d)(2S_x\mu_d + S_d(\mu_g + \mu_d)) + 2S_xS_d(\mu_g^2 - \mu_d^2) < (1 + \mu_d)^2. \quad (23)$$

The threshold on the total number of non-zero coefficients,  $S_g$ , is derived using (23), and can be found in [17, Corollary 3].

### 3.3.2. Condition for Recovering the $ML_0$ Solution using a Greedy Algorithm

We propose a greedy pursuit algorithm that can be used to recover the  $ML_0$  solution from (2). The proposed COMB-OMP algorithm follows a procedure similar to the OMP algorithm [24] and is presented in Table 1. The stopping criterion for this algorithm is either the maximum number of iterations/non-zero coefficients,  $T$ , or the  $\ell_2$  norm of the residual,  $\epsilon$ . In the algorithm,  $\pi(i)$  denotes the correlations computed for the current residual with the normalized atom  $\mathbf{g}_i$ . When updating the index set of chosen dictionary atoms,  $\mathcal{G}_t$ , we consider only the positive maximum correlation for atoms corresponding to  $\mathbf{X}$  and absolute maximum correlation for atoms corresponding to  $\mathbf{D}$ . This is consistent with our combined representation scheme. The solution update can be performed using a constrained least squares procedure. The final debiasing step computes the solution using the LSI algorithm described in Section 3.1. This step will be ignored when deriving the sparsity threshold, since it improves the solution only when the sparsity threshold is not satisfied and when there is additive noise in the combined model (2).

Table 1: The COMB-OMP Algorithm for Greedy Pursuit of a Combined Representation.

<b>Goal</b>
Recover the $ML_0$ solution from $\mathbf{y} = \mathbf{G}\boldsymbol{\delta}$ such that $\mathbf{I}_{\bar{\mathcal{X}}}\boldsymbol{\delta} \geq \mathbf{0}$ .
<b>Input</b>
$\mathbf{y}$ , the input vector.
$\mathbf{G} = [\mathbf{X} \quad \mathbf{D}]$ , the combined dictionary.
$T$ , the desired number of iterations.
$\epsilon$ , error tolerance.
<b>Initialization</b>
- Iteration count, $t = 0$ .
- Solution, $\boldsymbol{\delta}_t = \mathbf{0}$ .
- Residual, $\mathbf{r}_t = \mathbf{y} - \mathbf{G}\boldsymbol{\delta}_t = \mathbf{y}$ .
- Active coefficient supports, $\mathcal{X}_t = \{\}$ , $\mathcal{D}_t = \{\}$ , $\mathcal{G}_t = \{\}$ .
- All coefficient supports, $\bar{\mathcal{X}} = \{i\}_{i=1}^{K_x}$ , $\bar{\mathcal{D}} = \{i\}_{i=K_x+1}^{K_g}$ , $\bar{\mathcal{G}} = \bar{\mathcal{X}} \cup \bar{\mathcal{D}}$ .
- Non-active coefficient supports, $\mathcal{X}_t^c = \bar{\mathcal{X}}$ , $\mathcal{D}_t^c = \bar{\mathcal{D}}$ , $\mathcal{G}_t^c = \bar{\mathcal{G}}$ .

### Algorithm

Loop while  $t \leq T$  OR  $\|\mathbf{r}_t\|_2 > \epsilon$

- **Compute correlations:**

$$\pi(i) = \frac{\mathbf{r}_t^T \mathbf{g}_i}{\|\mathbf{g}_i\|_2} \text{ for } 1 \leq i \leq K_g.$$

- **Update support:**

$$\hat{i} = \underset{i \in \mathcal{X}_t^c}{\operatorname{argmax}} [\pi(i)]^+.$$

$$\hat{j} = \underset{j \in \mathcal{D}_t^c}{\operatorname{argmax}} |\pi(j)|.$$

$$\hat{k} = \underset{}{\operatorname{argmax}} ([\pi(\hat{i})]^+, |\pi(\hat{j})|).$$

If  $\hat{k} \in \mathcal{X}_t^c$ , then  $\mathcal{X}_{t+1} = \mathcal{X}_t \cup \{\hat{k}\}$ , else  $\mathcal{D}_{t+1} = \mathcal{D}_t \cup \{\hat{k}\}$ .

$$\mathcal{G}_{t+1} = \mathcal{X}_{t+1} \cup \mathcal{D}_{t+1}.$$

- **Update solution:**

$$\boldsymbol{\delta}_{t+1} = \underset{\boldsymbol{\delta}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{G}\boldsymbol{\delta}\|_2 \text{ subject to } \operatorname{support}(\boldsymbol{\delta}) = \mathcal{G}_{t+1}, \mathbf{I}_{\mathcal{X}_t} \boldsymbol{\delta} \geq \mathbf{0}.$$

- **Update residual:**  $\mathbf{r}_{t+1} = \mathbf{y} - \mathbf{G}\boldsymbol{\delta}_{t+1}$ .

- **Update support sets:**

$$\mathcal{G}_{t+1}^c = \bar{\mathcal{G}} - \mathcal{G}_{t+1}, \mathcal{X}_{t+1}^c = \bar{\mathcal{X}} - \mathcal{X}_{t+1}, \mathcal{D}_{t+1}^c = \bar{\mathcal{D}} - \mathcal{D}_{t+1}.$$

- **Update iteration count:**  $t = t + 1$ .

end

Debias to compute final  $\boldsymbol{\delta}$ :

$$\boldsymbol{\delta}_t = \underset{\boldsymbol{\delta}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{G}\boldsymbol{\delta}\|_2 \text{ subject to } \operatorname{support}(\boldsymbol{\delta}) = \mathcal{G}_t, \mathbf{I}_{\bar{\mathcal{X}}} \boldsymbol{\delta} \geq \mathbf{0}.$$

The sufficient sparsity threshold on the coefficient vector under which the COMB-OMP will recover the  $ML_0$  solution will be investigated. Some of the strategies used in the proofs are inspired by similar techniques used in [17, 18, 24]. In order to derive the threshold, we will divide the dictionary  $\mathbf{G} = [\mathbf{X} \ \mathbf{D}]$  into four sub-dictionaries  $\mathbf{X}_1 \in \mathbb{R}^{M \times S_x}$ ,  $\mathbf{X}_2 \in \mathbb{R}^{M \times (K_x - S_x)}$ ,  $\mathbf{D}_1 \in \mathbb{R}^{M \times S_d}$  and  $\mathbf{D}_2 \in \mathbb{R}^{M \times (K_d - S_d)}$ . We assume that the matrix  $\mathbf{G}_1 = [\mathbf{X}_1 \ \mathbf{D}_1]$  contains the atoms that participate in the representation and  $\mathbf{G}_2 = [\mathbf{X}_2 \ \mathbf{D}_2]$  contains those that do not participate. This implies that the signal  $\mathbf{y}$  can be represented as

$$\mathbf{y} = \mathbf{X}_1 \boldsymbol{\alpha}_1 + \mathbf{D}_1 \boldsymbol{\beta}_1, \quad (24)$$

where the elements of  $\boldsymbol{\alpha}_1 \in \mathbb{R}^{S_x}$  are strictly positive and those of  $\boldsymbol{\beta}_1 \in \mathbb{R}^{S_d}$  are non-zero.

**Lemma 3.4.** *When the matrix  $\mathbf{G}_1 = [\mathbf{X}_1 \ \mathbf{D}_1]$  has full column rank,  $\mathbf{y}$  is given by (24), and the residual  $\mathbf{r}_t$  of COMB-OMP satisfies*

$$\max(\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_1^T \mathbf{r}_t\|_\infty) = \|\mathbf{G}_1^T \mathbf{r}_t\|_\infty, \quad (25)$$

*the sufficient condition for COMB-OMP to uniquely recover the  $ML_0$  solution from (2) is*

$$\max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^\dagger \mathbf{g}_i\|_1 < 1. \quad (26)$$

PROOF. See Appendix A.

Note that the sufficient condition (26) given in Lemma 3.4 is the same for a general representation also [17, 24]. However, the important difference in the case of recovery using COMB-OMP is that (26) becomes sufficient only when (25) holds. As we will see in the following lemmas, this will lead to a significant difference in terms of sparsity threshold when compared to COMB-BP. We will first derive conditions under which the first step of the COMB-OMP, when  $\mathbf{y} = \mathbf{r}_0$ , will satisfy (25). This will then be extended to the residuals at all steps,  $\mathbf{r}_t$ , where  $t \geq 1$ .

**Lemma 3.5.** *When the matrix  $\mathbf{G}_1 = [\mathbf{X}_1 \ \mathbf{D}_1]$  has full column rank, and  $\mathbf{y}$  is given as (24), (25) will be satisfied for  $\mathbf{y} = \mathbf{r}_0$  if*

$$(S_x - 1)\mu_d + S_d\mu_g < \frac{1}{2}. \quad (27)$$

PROOF. See Appendix B.

The condition given by (27) needs to be satisfied even if there is one non-negative component in the combined representation. For now, let us assume that (25) holds for all  $\mathbf{r}_t$ , where  $t \geq 1$ , and derive the threshold on  $S_g$  such that the  $ML_0$  solution can be recovered from (2). It will be shown later in the section that the threshold on  $S_g$  obtained indeed implies that (25) holds for all  $\mathbf{r}_t, t \geq 1$ .

**Lemma 3.6.** *When the matrix  $\mathbf{G}_1$  has full column rank, and  $\mathbf{y}$  is given as (24), the sufficient condition for (26) to be satisfied is*

$$\frac{S_x\mu_d + S_d\mu_g}{1 - (S_x\mu_d + S_d\mu_g - \mu_d)} < 1 \quad (28)$$

PROOF. See Appendix C.

When (28) is satisfied, (27) holds as well. Since the number of non-zero coefficients,  $S_x$  and  $S_d$ , of  $\alpha_1$  and  $\beta_1$  are unknown, we need to derive the condition on recovery that depends only on the number of non-zero coefficients of the combined representation  $S_g$ .

**Lemma 3.7.** *When the matrix  $\mathbf{G}_1$  has full column rank, and  $\mathbf{y}$  is given as (24), the sparsity threshold on the combined coefficient vector  $\delta$  for (28) to hold is*

$$S_g < 0.5 \left( 1 + \frac{1}{\mu_g} \right). \quad (29)$$

PROOF. See Appendix D.

A similar procedure to obtain the threshold on  $S_g$  using (27), results in  $S_g < 0.5(2 + 1/\mu_g)$ , which is only slightly better than (29). Moreover, as stated already (28) implies (27) and not vice-versa. Therefore the sparsity threshold in (29) cannot be made better. We will assume that  $\mu_g = \mu_d$ , as obtained in the proof of the above lemma, and show that the above bound is sufficient for recovering the subsequent atoms using the residuals  $\mathbf{r}_t$ , when  $t \geq 1$ .

**Lemma 3.8.** *When the matrix  $\mathbf{G}_1$  has full column rank, and  $\mathbf{y}$  is given as (24), (25) will hold true for residual at any step,  $\mathbf{r}_t$  for  $t \geq 1$ , when  $\mu_g = \mu_d$  and (29) is satisfied.*

PROOF. See Appendix E.

Now, we are ready to state our main theorem without proof, since it follows directly from the lemmas stated in this section.

**Theorem 3.9.** *For any  $\mathbf{y}$  that follows the combined model in (2), COMB-OMP will recover the  $ML_0$  solution if the number of non-zero coefficients,  $S_g$ , is less than  $0.5(1 + 1/\mu_g)$ .*

From the lemmas proved in this section, it is clear that this threshold cannot be made better. Contrast this with the case of recovery using a convex program discussed in Section 3.3.1, as well as sparsity thresholds for recovery using BP and OMP when  $\alpha$  and  $\beta$  are general sparse vectors [17, Eqn. (13)]. Figure 1 compares the thresholds on  $S_g$  when  $\mu_g = 0.01$  and  $\mu_d$  varies from 0 to 0.01. We can see that an improvement up to a

factor of two can be observed with COMB-BP, BP and OMP algorithms when compared to COMB-OMP. From the proof of Lemma 3.5, it can be observed that introducing non-negative constraint on even one coefficient in the representation drastically alters the deterministic sparsity threshold of greedy OMP-like algorithms. Note that, however, the sparsity thresholds are pessimistic and in the experiments provided in the next section, COMB-OMP performs better than OMP and also has much reduced computational complexity when compared to COMB-BP and BP.

#### 4. Experiments

The COMB-BP and the COMB-OMP algorithms incorporate the prior knowledge that a set of coefficients to be recovered are non-negative. If we use BP and OMP algorithms for recovery, this prior knowledge cannot be exploited. In order to establish that this additional information leads to improvement in recovery performance, we performed numerical experiments by realizing the elements of dictionary  $\mathbf{G}$  from an i.i.d. zero-mean, unit-variance, Gaussian distributions. The non-zero coefficients in the coefficient vector  $\boldsymbol{\delta}$  are either signs ( $\pm 1$ ) or realized from a uniform distribution. We varied the proportion of the non-negative and the unconstrained coefficients in the combined representation and tested the performance of COMB-BP, BP, COMB-OMP and OMP algorithms in exact and approximate recovery. For the case of exact recovery, we also compared NN-BP and NN-OMP algorithms which impose the constraint that all coefficients are non-negative.

The total number of atoms in  $\mathbf{G}$  was fixed at  $K_g = 200$ . The three cases tested were (a)  $K_x = 50$ ,  $K_d = 150$ , (b)  $K_x = 100$ ,  $K_d = 100$ , and (c)  $K_x = 150$ ,  $K_d = 50$ . The location of the non-zero coefficients in  $\boldsymbol{\delta}$  were fixed uniformly at random. When the non-zero coefficients were realized from a uniform distribution, the non-negative coefficients were obtained from the uniform distribution  $U(0, 1)$  and the general coefficients were obtained from  $U(-1, 1)$ . When the non-zero coefficients were signs, the general coefficients were obtained with equiprobable positive and negative signs. The number of non-zero coefficients  $S_x$  and  $S_d$  were varied from 1 to 30 each, and hence the total number of non-zero coefficients,  $S_g$ , varied from 2 to 60.  $\mathbf{y}$  was obtained using the combined model (2), and coefficient recovery was performed using the six algorithms. The relative recovery error

between the recovered coefficient vector  $\hat{\boldsymbol{\delta}}$  and the actual coefficient vector  $\boldsymbol{\delta}$  is

$$RRE = \frac{\|\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}\|_2^2}{\|\boldsymbol{\delta}\|_2^2}. \quad (30)$$

If the  $RRE$  is less than  $10^{-6}$ , the coefficient is said to be recovered exactly. Each experiment was repeated 10 times in order to measure the average performance in cases of both exact and approximate recovery. The average relative recovery error is the mean of  $RRE$  over all iterations.

Let us first consider the case when the coefficients are realized from the uniform distribution. From Figure 2 it can be seen that as  $K_x$  increases, the performance of COMB-BP and COMB-OMP become increasingly better when compared to that of BP and OMP respectively. In particular, the performance of COMB-BP substantially improves as the non-negative component in the representation becomes bigger. Note that the recovery performance of NN-OMP and NN-BP algorithms do not compare well with the rest of the algorithms, since they impose the constraint that all coefficients are non-negative, whereas actually only a part of them are. Furthermore, in Figure 3, it can be observed that COMB-BP and COMB-OMP algorithms exhibit lesser average RRE when compared to their unconstrained counterparts. In this case, the gap in performance between OMP and COMB-OMP is very pronounced when  $K_x$  is large. Similar behavior can be observed when non-zero coefficients are signs (Figures 4 and 5) but in general the differences in performance of the algorithms are less prominent. The experiments clearly show that COMB-BP and COMB-OMP perform better than BP and OMP respectively, although the sparsity thresholds derived in the previous section did not point to such an improvement. This is because the deterministic sparsity bounds are generally pessimistic. Furthermore, the presence of a large non-negative component substantially improves the recovery performances of COMB-BP and COMB-OMP. The average RRE using NN-OMP and NN-BP algorithms are not shown since they are much higher than those of the other algorithms.

Finally, the average CPU time taken by each of the algorithms to recover a coefficient vector is computed and shown in Figure 6, for the case when  $\mathbf{G}$  is obtained from a Gaussian ensemble, the non-zero coefficients are realized from a uniform distribution,  $K_x = 100$  and  $K_d = 100$ . The experiments were performed using MATLAB R2010b on a 2.8GHz, Intel i7 desktop. Clearly, COMB-OMP and OMP have much lesser computa-



tional complexity when compared to COMB-BP and BP.

## 5. Conclusions

We considered the problem of recovering sparse solutions from an overcomplete linear model when the solution vector was constrained to be either completely or partially non-negative. When the solution was completely non-negative, based on the theory of polytopes, we derived conditions on the dictionary for the existence of a unique solution. In the case of combined sparse representations, we considered cases when the coefficient support was completely known, partially known or completely unknown. When the coefficient support was completely unknown, we proposed the COMB-OMP algorithm and derived the deterministic sparsity threshold that guarantees recovery of the unique, minimum  $\ell_0$  norm solution. Experimental results, using dictionaries drawn from a Gaussian ensemble and non-zero coefficients realized from a uniform distribution or an equiprobable distribution of signs, showed that the COMB-BP and COMB-OMP algorithms perform better in terms of exact and approximate recovery compared to their unconstrained counterparts. A possible direction for future work is to derive probabilistic sparsity thresholds for NN-BP and COMB-BP algorithms, under appropriate assumptions on the dictionary and the coefficient vectors, that will explain the improved experimental performance of sparse recovery algorithms with non-negativity constraints when compared to their unconstrained versions.

## Appendix A. Proof of Lemma 3.4

For COMB-OMP to recover the unique sparsest representation, no atom from  $\mathbf{G}_2$  must enter the support set  $\mathcal{G}_t$  at any iteration. Therefore, the residual  $\mathbf{r}_t$  at each iteration  $t$  must satisfy the condition

$$\rho(\mathbf{r}_t) \equiv \frac{\max(\max(\mathbf{X}_2^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_2^T \mathbf{r}_t\|_\infty)}{\max(\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_1^T \mathbf{r}_t\|_\infty)} < 1. \quad (\text{A.1})$$

Since

$$\max(\max(\mathbf{X}_2^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_2^T \mathbf{r}_t\|_\infty) \leq \|\mathbf{G}_2^T \mathbf{r}_t\|_\infty,$$

$\rho(\mathbf{r}_t)$  can be bounded as

$$\rho(\mathbf{r}_t) \leq \frac{\|\mathbf{G}_2^T \mathbf{r}_t\|_\infty}{\max(\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_1^T \mathbf{r}_t\|_\infty)} \quad (\text{A.2})$$

$$= \frac{\|\mathbf{G}_2^T (\mathbf{G}_1^\dagger)^T \mathbf{G}_1^T \mathbf{r}_t\|_\infty}{\max(\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_1^T \mathbf{r}_t\|_\infty)} \quad (\text{A.3})$$

$$\leq \frac{\|\mathbf{G}_2^T (\mathbf{G}_1^\dagger)^T\|_{\infty, \infty} \|\mathbf{G}_1^T \mathbf{r}_t\|_\infty}{\max(\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}), \|\mathbf{D}_1^T \mathbf{r}_t\|_\infty)} \quad (\text{A.4})$$

$$= \|\mathbf{G}_2^T (\mathbf{G}_1^\dagger)^T\|_{\infty, \infty} \quad (\text{A.5})$$

$$= \|\mathbf{G}_1^\dagger \mathbf{G}_2\|_{1,1} \quad (\text{A.6})$$

$$= \max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^\dagger \mathbf{g}_i\|_1 \quad (\text{A.7})$$

Eqn. (A.3) holds since  $(\mathbf{G}_1^\dagger)^T \mathbf{G}_1^T$  is an orthoprojector onto the column space of  $\mathbf{G}_1$ . Both  $\mathbf{y}$  and  $\mathbf{G}_2 \boldsymbol{\delta}_t$  lie in the column space of  $\mathbf{G}_1$  and hence  $\mathbf{r}_t$  lies in the same space. The properties of  $\|\cdot\|_{\infty, \infty}$  ensures that (A.4) is true. By assumption, the denominator of (A.4) equals  $\|\mathbf{G}_1^T \mathbf{r}_t\|_\infty$ . Therefore, (A.5) holds true and (A.6) follows from relation  $\|\mathbf{A}^T\|_{\infty, \infty} = \|\mathbf{A}\|_{1,1}$  for any matrix  $\mathbf{A}$ . From (A.1), (A.2) and (A.7), the sufficient condition provided in (26) is obtained.

## Appendix B. Proof of Lemma 3.5

For (25) to be satisfied, the sufficient condition is that

$$\max(\mathbf{X}_1^T \mathbf{r}_t, \mathbf{0}) = \|\mathbf{X}_1^T \mathbf{r}_t\|_\infty. \quad (\text{B.1})$$

Therefore, we only have to consider the case where an atom from  $\mathbf{X}_1$  will be picked. Let us denote  $\boldsymbol{\delta}_1 = [\boldsymbol{\alpha}_1^T \ \boldsymbol{\beta}_1^T]^T$  and  $\mathbf{z} = \mathbf{X}_1^T \mathbf{y} = \mathbf{X}_1^T \mathbf{G}_1 \boldsymbol{\delta}_1$ . We will derive the bounds on the maximum positive value,  $z_m$ , and the minimum negative value,  $z_n$ , of  $\mathbf{z}$ . We denote the smallest possible lower bound on  $z_m$  as  $\hat{z}_m$ , and the largest possible lower bound on  $|z_n|$  as  $\hat{z}_n$ . The worst-case guarantee for (B.1) to be true is

$$\hat{z}_m > \hat{z}_n. \quad (\text{B.2})$$

Using the fact that

$$\mathbf{X}_1^T \mathbf{G}_1 = [\mathbf{I}_{S_x} \ \mathbf{0}] + [\mathbf{X}_1^T \mathbf{X}_1 - \mathbf{I}_{S_x} \ \mathbf{X}_1^T \mathbf{D}_1],$$

the correlation vector  $\mathbf{z}$  can be expressed as

$$\mathbf{z} = \boldsymbol{\alpha}_1 + [\mathbf{X}_1^T \mathbf{X}_1 - \mathbf{I}_{S_x} \quad \mathbf{X}_1^T \mathbf{D}_1] \boldsymbol{\delta}_1. \quad (\text{B.3})$$

Let us define  $\mathbf{C}_1 = [\mathbf{X}_1^T \mathbf{X}_1 - \mathbf{I}_{S_x} \quad \mathbf{X}_1^T \mathbf{D}_1]$  and the elementwise bounds on the submatrices are,

$$\begin{aligned} |\mathbf{X}_1^T \mathbf{X}_1 - \mathbf{I}_{S_x}| &\leq \mu_x(\mathbf{1}_{S_x, S_x} - \mathbf{I}_{S_x}) \\ &\leq \mu_d(\mathbf{1}_{S_x, S_x} - \mathbf{I}_{S_x}), \end{aligned}$$

and  $\mathbf{X}_1^T \mathbf{D}_1 \leq |\mu_g \mathbf{1}_{S_x, S_d}|$ . It is clear that the maximum row sum of  $\mathbf{C}_1$  is

$$\|\mathbf{C}_1\|_{\infty, \infty} \leq (S_x - 1)\mu_d + S_d\mu_g. \quad (\text{B.4})$$

In order to derive the smallest lower bound on  $z_m$ , we will assume that all the coefficients in  $\boldsymbol{\delta}_1$  have the same absolute value given by  $\alpha$ , and hence, from (B.3), we have

$$\begin{aligned} z_m &\geq \alpha - \alpha \|\mathbf{C}_1\|_{\infty, \infty} \\ &\geq \alpha(1 - [(S_x - 1)\mu_d + S_d\mu_g]) \equiv \hat{z}_m. \end{aligned} \quad (\text{B.5})$$

The required bound on  $z_n$  can be obtained by setting one element of  $\boldsymbol{\alpha}_1$  as  $\hat{\alpha}$ , where  $0 < \hat{\alpha} < \alpha$ , and the absolute value of all the other elements of  $\boldsymbol{\delta}_1$  as  $\alpha$ . We now have

$$z_n \geq \hat{\alpha} - \alpha \|\mathbf{C}_1\|_{\infty, \infty},$$

and as  $\hat{\alpha} \rightarrow 0$ ,

$$|z_n| < \alpha[(S_x - 1)\mu_d + S_d\mu_g] \equiv \hat{z}_n, \quad (\text{B.6})$$

which is the largest possible lower bound on  $z_n$ . Substituting (B.5) and (B.6) in (B.2), results in (27).

### Appendix C. Proof of Lemma 3.6

The condition for success of COMB-OMP can be written as

$$\max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^\dagger \mathbf{g}_i\|_1 \leq \|(\mathbf{G}_1^T \mathbf{G}_1)^{-1}\|_{1,1} \max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^T \mathbf{g}_i\|_1, \quad (\text{C.1})$$

using the property of  $\|\cdot\|_{1,1}$  and the fact that  $\mathbf{G}_1^\dagger = (\mathbf{G}_1^T \mathbf{G}_1)^{-1} \mathbf{G}_1^T$ .

In order to compute the lower bound for  $\|(\mathbf{G}_1^T \mathbf{G}_1)^{-1}\|_{1,1}$ , we first expand the Gramm matrix

$$\mathbf{G}_1^T \mathbf{G}_1 = \mathbf{I}_{S_g} + \mathbf{C}, \quad (\text{C.2})$$

where

$$\mathbf{C} \equiv \begin{bmatrix} \mathbf{X}_1^T \mathbf{X}_1 - \mathbf{I}_{S_x} & \mathbf{X}_1^T \mathbf{D}_1 \\ \mathbf{D}_1^T \mathbf{X}_1 & \mathbf{D}_1^T \mathbf{D}_1 - \mathbf{I}_{S_d} \end{bmatrix}.$$

$\mathbf{C}$  can be bounded elementwise as

$$\begin{aligned} |\mathbf{C}| &\leq \begin{bmatrix} \mu_x(\mathbf{1}_{S_x, S_x} - \mathbf{I}_{S_x}) & \mu_g \mathbf{1}_{S_x, S_d} \\ \mu_g \mathbf{1}_{S_d, S_x} & \mu_d(\mathbf{1}_{S_d, S_d} - \mathbf{I}_{S_d}) \end{bmatrix} \\ &\leq \begin{bmatrix} \mu_d(\mathbf{1}_{S_x, S_x} - \mathbf{I}_{S_x}) & \mu_g \mathbf{1}_{S_x, S_d} \\ \mu_g \mathbf{1}_{S_d, S_x} & \mu_d(\mathbf{1}_{S_d, S_d} - \mathbf{I}_{S_d}) \end{bmatrix}, \end{aligned}$$

since  $\mu_x \leq \mu_d$  by assumption. The maximum column sum of  $\mathbf{C}$  is bounded as

$$\|\mathbf{C}\|_{1,1} \leq (S_x - 1)\mu_d + S_d\mu_g, \quad (\text{C.3})$$

since  $S_x \leq S_d$ . From (C.2), we also observe that  $\|\mathbf{C}\|_{1,1} < 1$ , since  $\mathbf{G}_1^T \mathbf{G}_1$  is strictly diagonally dominant, because of the linear independence of the columns of  $\mathbf{G}_1$ . Using (C.2), we can write

$$\|(\mathbf{G}_1^T \mathbf{G}_1)^{-1}\|_{1,1} = \|(\mathbf{I}_{S_g} + \mathbf{C})^{-1}\|_{1,1}. \quad (\text{C.4})$$

The Neumann series  $\sum_k (-\mathbf{C})^k$  converges to  $(\mathbf{I}_{S_g} + \mathbf{C})^{-1}$ , whenever  $\|\mathbf{C}\|_{1,1} < 1$  [25] and hence (C.4) can be expressed as

$$\begin{aligned} \|(\mathbf{G}_1^T \mathbf{G}_1)^{-1}\|_{1,1} &= \left\| \sum_{k=1}^{\infty} (-\mathbf{C})^k \right\|_{1,1} \\ &\leq \sum_{k=1}^{\infty} \|\mathbf{C}\|_{1,1}^k \\ &= \frac{1}{1 - \|\mathbf{C}\|_{1,1}} \\ &\leq \frac{1}{1 - (S_x\mu_d + S_d\mu_g - \mu_d)}, \end{aligned} \quad (\text{C.5})$$

using (C.3). If  $\mathbf{g}_i$  is a vector chosen from  $\mathbf{X}_2$ ,  $|\mathbf{G}_1^T \mathbf{g}_i| \leq [\mu_d \mathbf{1}_{S_x}^T \quad \mu_g \mathbf{1}_{S_d}^T]^T$  and hence we have

$$\max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^T \mathbf{g}_i\|_1 \leq S_x\mu_d + S_d\mu_g = (S_x + S_d)\mu_d + S_d(\mu_g - \mu_d). \quad (\text{C.6})$$

If  $\mathbf{g}_i$  is chosen from  $\mathbf{D}_2$ ,  $|\mathbf{G}_1^T \mathbf{g}_i| \leq [\mu_g \mathbf{1}_{S_x}^T \quad \mu_d \mathbf{1}_{S_d}^T]^T$ . Therefore

$$\max_{i \in \mathcal{G}^c} \|\mathbf{G}_1^T \mathbf{g}_i\|_1 \leq S_x \mu_g + S_d \mu_d = (S_x + S_d) \mu_d + S_x (\mu_g - \mu_d). \quad (\text{C.7})$$

Since  $S_d \geq S_x$ , among (C.6) and (C.7), we will choose (C.6) as our bound. Substituting (C.6) and (C.5) in (C.1), we can obtain (28) as the condition for COMB-OMP to succeed.

#### Appendix D. Proof of Lemma 3.7

Rewriting (28), we obtain

$$S_x < \frac{1 + \mu_d - 2S_d \mu_g}{2\mu_d} \equiv \psi(S_d).$$

The threshold on the total number of non-zero coefficients,  $S_g$ , can be obtained by minimizing  $\psi(S_d) + S_d$  over  $S_d$ . The constraint is that  $S_d \geq 1$  since  $S_x \leq S_d$  and the overall representation will have at least one non-zero coefficient. Denoting  $S$  to be the sparsity threshold, it can be obtained as

$$S = \min_{S_d \geq 1} [S_d + \psi(S_d)].$$

Relaxing the constraint that  $S_d$  is an integer, the minimum will be obtained when  $\mu_g = \mu_d$  and the minimum value is

$$S = 0.5 \left( 1 + \frac{1}{\mu_g} \right),$$

which is the strict upper bound on  $S_g$ .

#### Appendix E. Proof of Lemma 3.8

Since we assumed that  $\mu_g = \mu_d$ , we will assume that the overall coherence of  $\mathbf{G}_1$  is  $\mu_g$  and the total number of columns in  $\mathbf{G}_1$  is  $S_g$ . We will denote  $\mathbf{G}_1 = [\mathbf{G}_a \quad \mathbf{G}_b]$ , where  $\mathbf{G}_a$  with  $S_a$  columns contains the atoms already chosen for the representation and  $\mathbf{G}_b$  contains  $S_b = S_g - S_a$  atoms, one of which will be chosen by the residual. The residual  $\mathbf{r}_t$  will be simply denoted as  $\mathbf{r}$  for notational convenience and is obtained using a least squares procedure as

$$\mathbf{r} = \mathbf{y} - \mathbf{G}_a \boldsymbol{\delta}_a, \quad (\text{E.1})$$

where

$$\boldsymbol{\delta}_a = \mathbf{G}_a^\dagger \mathbf{y}. \quad (\text{E.2})$$

Let us denote the correlation vector as

$$\mathbf{z} = \mathbf{G}_b^T \mathbf{r}. \quad (\text{E.3})$$

Similar to the proof of Lemma 3.5, we are only concerned about recovering the non-negative coefficients as it will give us sufficient conditions under which (25) will be satisfied. The bounds on the maximum positive value,  $z_m$ , and the minimum negative value,  $z_n$ , of  $\mathbf{z}$  will be derived assuming that all the elements in  $\boldsymbol{\delta}_b$  are constrained to be non-negative. In this case, the smallest possible lower bound on  $z_m$ , given by  $\hat{z}_m$  and the largest possible lower bound on  $|z_n|$  given by  $\hat{z}_n$ . For (25) to hold for any  $\mathbf{r}_t$ , where  $t \geq 1$ , similar to the proof of Lemma 3.5, we need to show that

$$\hat{z}_m > \hat{z}_n. \quad (\text{E.4})$$

We will first expand (E.3) using (E.1) and (E.2)

$$\begin{aligned} \mathbf{z} &= \mathbf{G}_b^T (\mathbf{y} - \mathbf{G}_a \mathbf{G}_a^\dagger \mathbf{y}) \\ &= \mathbf{G}_b^T (\mathbf{I} - \mathbf{G}_a \mathbf{G}_a^\dagger) \mathbf{G}_1 \boldsymbol{\delta}_1 \end{aligned} \quad (\text{E.5})$$

$$= \mathbf{G}_b^T (\mathbf{I} - \mathbf{G}_a \mathbf{G}_a^\dagger) \mathbf{G}_b \boldsymbol{\delta}_b, \quad (\text{E.6})$$

which is obtained by substituting  $\mathbf{G}_1 = [\mathbf{G}_a \quad \mathbf{G}_b]$  and  $\boldsymbol{\delta}_1 = [\boldsymbol{\delta}_a^T \quad \boldsymbol{\delta}_b^T]^T$  in (E.5). Let us denote any two distinct columns from  $\mathbf{G}_b$  by  $\mathbf{g}_i$  and  $\mathbf{g}_j$ . Let us also denote the matrix  $\mathbf{Q} = \mathbf{G}_b^T (\mathbf{I} - \mathbf{G}_a \mathbf{G}_a^\dagger) \mathbf{G}_b$ , which is of size  $S_b \times S_b$ , and designate its  $(i, j)^{\text{th}}$  element to be  $q_{ij}$ . The correlation vector in (E.6) can be now expressed as

$$\mathbf{z} = \mathbf{Q} \boldsymbol{\delta}_b$$

We will compute bounds on the diagonal and off-diagonal elements of  $\mathbf{Q}$ . Expanding  $\mathbf{G}^\dagger$ , we can write

$$|q_{ij}| = |\mathbf{g}_i^T [\mathbf{I} - \mathbf{G}_a (\mathbf{G}_a^T \mathbf{G}_a)^{-1} \mathbf{G}_a^T] \mathbf{g}_j| \quad (\text{E.7})$$

$$\leq |\mathbf{g}_i^T \mathbf{g}_j| + |\mathbf{g}_i^T \mathbf{G}_a (\mathbf{G}_a^T \mathbf{G}_a)^{-1} \mathbf{G}_a^T \mathbf{g}_j| \quad (\text{E.8})$$

$$\leq |\mathbf{g}_i^T \mathbf{g}_j| + \|\mathbf{G}_a^T \mathbf{g}_i\|_2^2 \|(\mathbf{G}_a^T \mathbf{G}_a)^{-1}\|_2. \quad (\text{E.9})$$

Eqn. (E.8) follows from applying triangle inequality on (E.7) and (E.9) is obtained by upper bounding the second term in the right hand side of (E.8). We can express

$$\|\mathbf{G}_a^T \mathbf{g}_i\|_2^2 \leq S_a \mu_g^2$$

since the maximum absolute coherence between any two elements in  $\mathbf{G}_1$  is  $\mu_g$ . Since  $\|(\mathbf{G}_a^T \mathbf{G}_a)^{-1}\|_2 \leq 1/\lambda_{\min}(\mathbf{G}_a^T \mathbf{G}_a)$ , and by Gershgorin's disc theorem [26, Theorem 6.1.1],  $\lambda_{\min}(\mathbf{G}_a^T \mathbf{G}_a) \leq [1 - \mu_g(S_a - 1)]^+$ , we can rewrite (E.9) as

$$|q_{ij}| \leq \mu_g + \frac{S_a \mu_g^2}{[1 - \mu_g(S_a - 1)]^+}. \quad (\text{E.10})$$

When  $i = j$ , we have

$$|q_{ii}| = |\mathbf{g}_i^T [\mathbf{I} - \mathbf{G}_a (\mathbf{G}_a^T \mathbf{G}_a)^{-1} \mathbf{G}_a^T] \mathbf{g}_i| \quad (\text{E.11})$$

$$\geq |\mathbf{g}_i^T \mathbf{g}_i| - |\mathbf{g}_i^T \mathbf{G}_a (\mathbf{G}_a^T \mathbf{G}_a)^{-1} \mathbf{G}_a^T \mathbf{g}_i| \quad (\text{E.12})$$

$$\geq 1 - \frac{S_a \mu_g^2}{[1 - \mu_g(S_a - 1)]^+}. \quad (\text{E.13})$$

Eqn. (E.12) follows from applying reverse triangle inequality on (E.11) and (E.13) is obtained by following steps similar to the derivation of upper bound on  $|q_{ij}|$ . The bounds given by (E.10) and (E.13) are valid only if  $1 - \mu_g(S_a - 1) > 0$ , which can be verified by substituting  $\mu_g < 1/(2S_g - 1)$ , from (29), and  $S_a < S_g$ . Therefore, (E.11) and (E.13) can be rewritten as  $|q_{ij}| \leq q_1$ ,  $|q_{ii}| \geq q_2$ , where

$$q_1 = C_1 \mu_g$$

$$q_2 = C_1 (1 - S_a \mu_g),$$

and  $C_1 = (1 + \mu_g)/[1 - \mu_g(S_a - 1)]$ .

We know that the diagonal elements of  $\mathbf{Q}$  are lower bounded by  $q_1$  and the off-diagonal elements are upper bounded by  $q_2$ . Since there are  $S_b$  rows in  $\mathbf{Q}$ , the smallest lower bound on  $z_m$  is

$$z_m \geq \alpha q_2 - \alpha(S_b - 1)q_1 \equiv \hat{z}_m \quad (\text{E.14})$$

which is obtained when all the elements in  $\boldsymbol{\delta}_b$  are set to  $\alpha$ . The required bound on  $z_n$  is obtained by setting one element of  $\boldsymbol{\delta}_b$  corresponding to a positive coefficient as  $\hat{\alpha}$ , where  $0 < \hat{\alpha} < \alpha$ ,  $\hat{\alpha}$  approaches zero and all the other values in  $\boldsymbol{\delta}_b$  are set to  $\alpha$ .  $z_n$  can be now bounded as  $z_n \geq \hat{\alpha} q_2 - \alpha q_1(S_b - 1)$ . As  $\hat{\alpha} \rightarrow 0$ , we have

$$|z_n| < \alpha(S_b - 1)q_2 \equiv \hat{z}_n. \quad (\text{E.15})$$

Using (E.4), (E.14) and (E.15), we have

$$\mu_g < \frac{1}{2S_g - S_a - 2},$$

which is always satisfied since we know from (29) that  $\mu_g < 1/(2S_g - 1)$  and  $S_a \geq 1$ .

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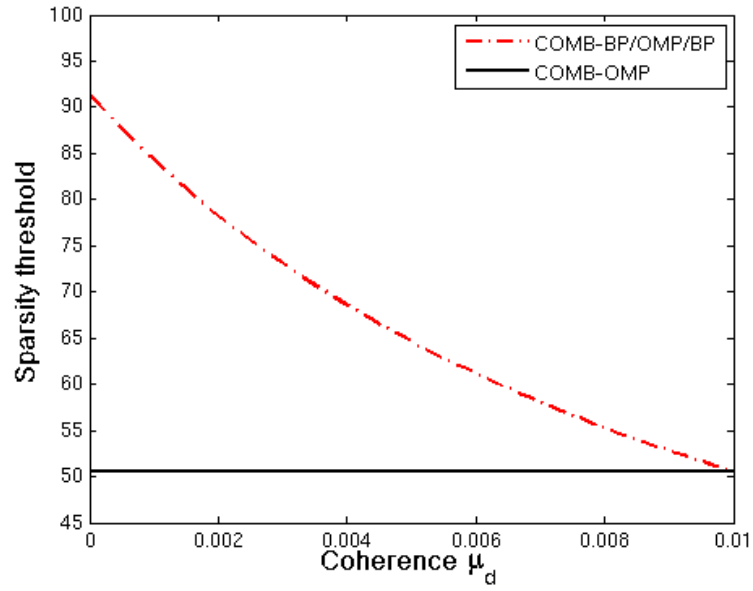


Figure 1: Deterministic sparsity thresholds for COMB-BP, BP, COMB-OMP and OMP with  $\mu_g = 0.01$  in order to recover the  $ML_0$  solution from (2).

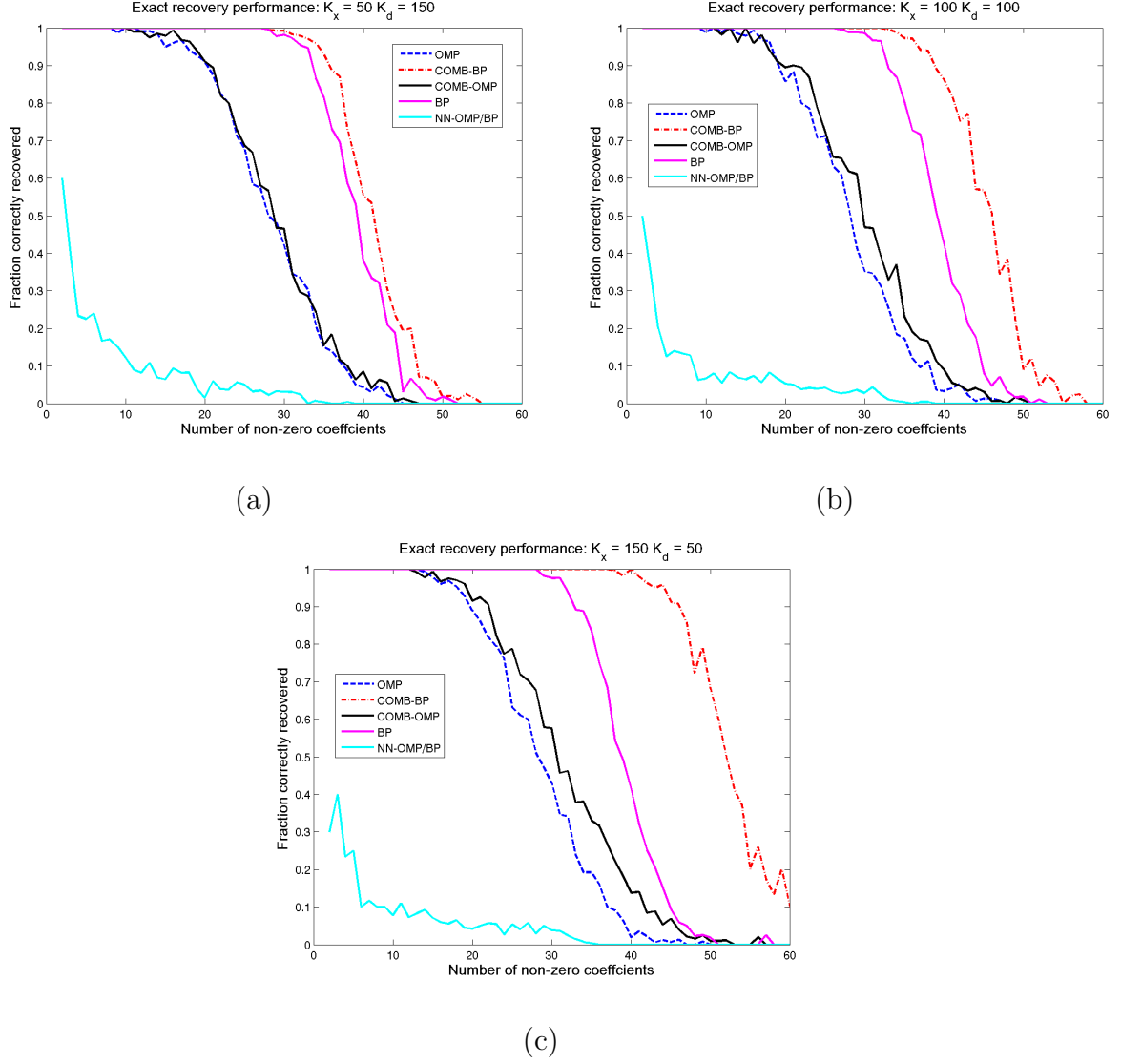
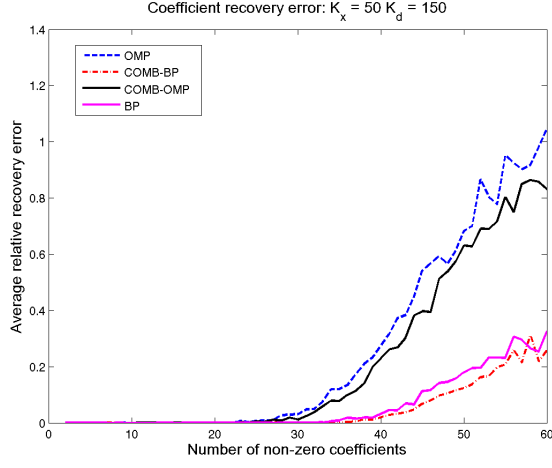
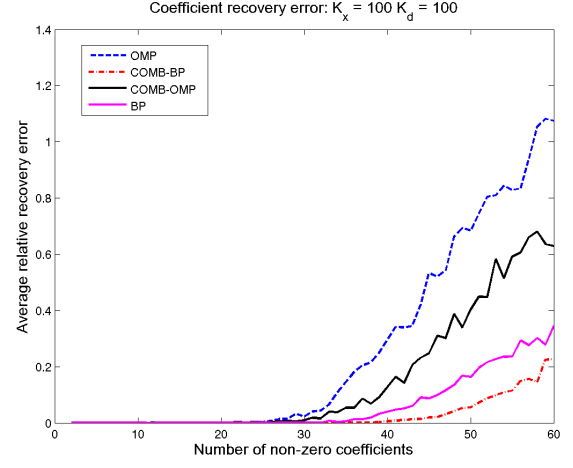


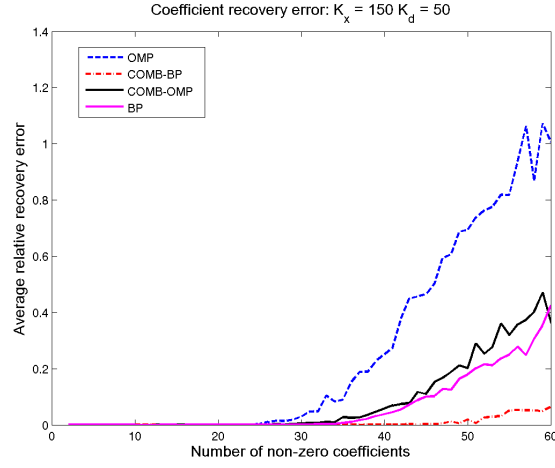
Figure 2: Exact recovery performance of COMB-BP, BP, COMB-OMP, OMP, NN-OMP and NN-BP when  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are realized from a uniform distribution. (a)  $K_x = 50$  and  $K_d = 150$ , (b)  $K_x = 100$  and  $K_d = 100$ , (c)  $K_x = 150$  and  $K_d = 50$ .



(a)

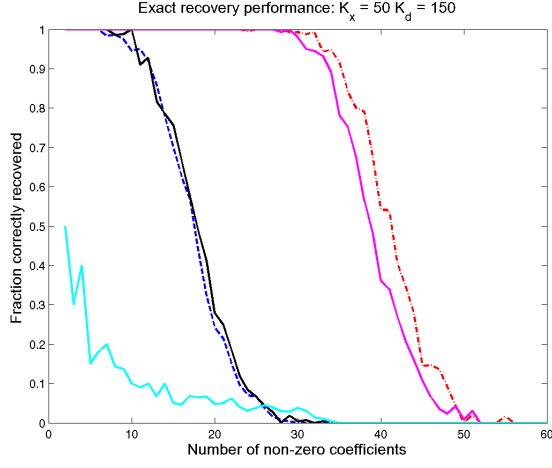


(b)

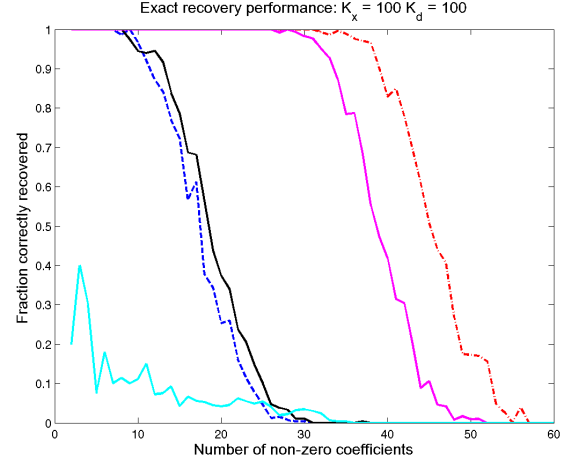


(c)

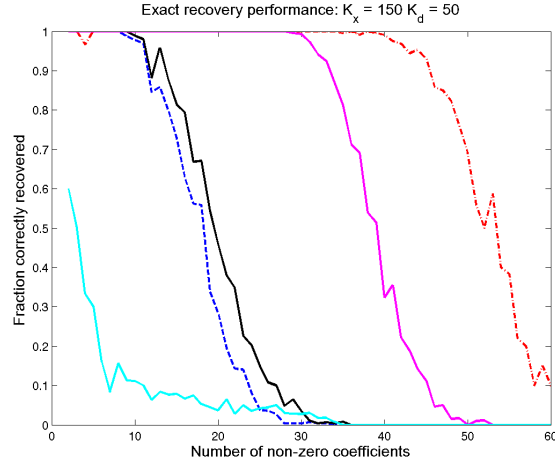
Figure 3: Average relative recovery error of COMB-BP, BP, COMB-OMP and OMP when  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are realized from a uniform distribution. (a)  $K_x = 50$  and  $K_d = 150$ , (b)  $K_x = 100$  and  $K_d = 100$ , (c)  $K_x = 150$  and  $K_d = 50$ .



(a)

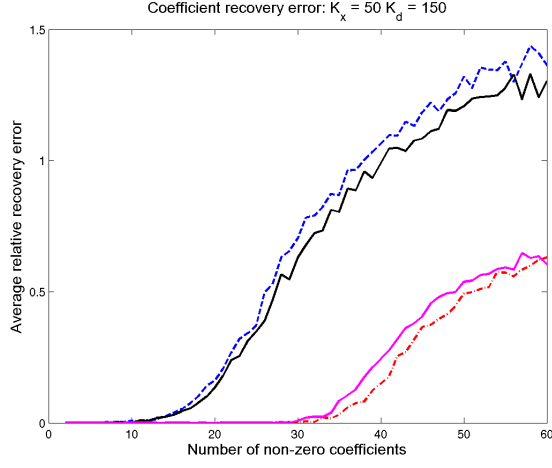


(b)

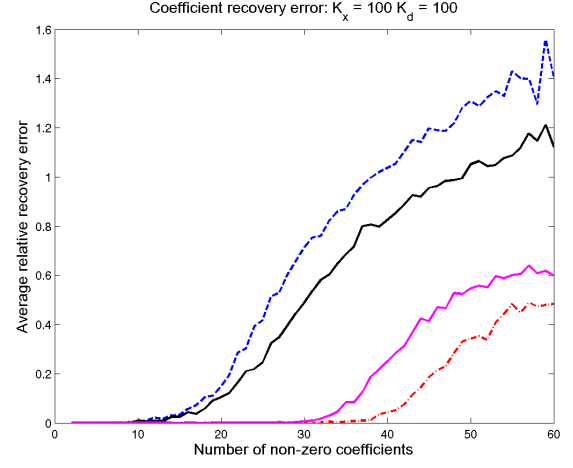


(c)

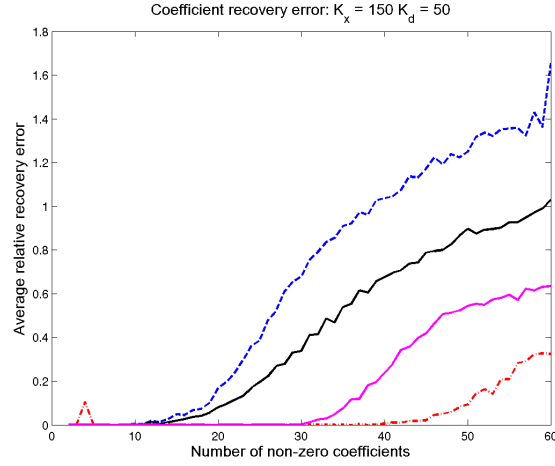
Figure 4: Exact recovery performance of COMB-BP, BP, COMB-OMP and OMP when  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are signs. (a)  $K_x = 50$  and  $K_d = 150$ , (b)  $K_x = 100$  and  $K_d = 100$ , (c)  $K_x = 150$  and  $K_d = 50$ . The legend here is same as that of Figure 2.



(a)



(b)



(c)

Figure 5: Average relative recovery error of COMB-BP, BP, COMB-OMP and OMP when  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are signs. (a)  $K_x = 50$  and  $K_d = 150$ , (b)  $K_x = 100$  and  $K_d = 100$ , (c)  $K_x = 150$  and  $K_d = 50$ . The legend here is same as that of Figure 3.

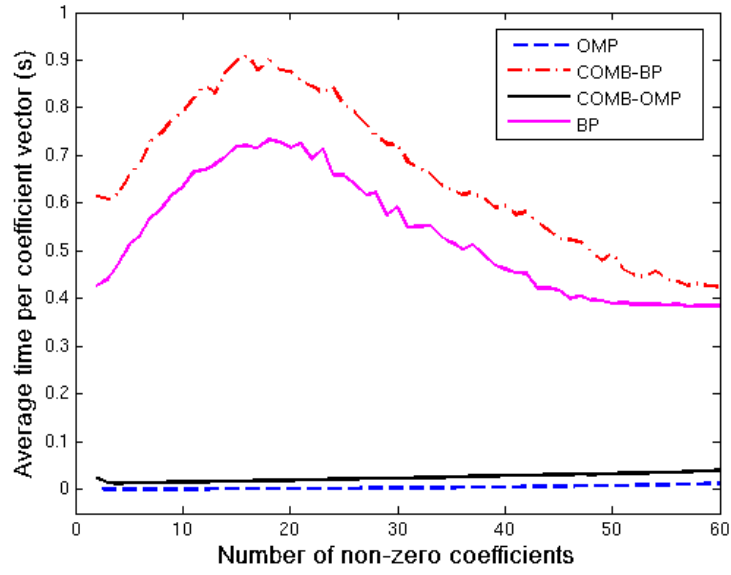


Figure 6: Average CPU time in seconds taken by COMB-BP, BP, COMB-OMP and OMP algorithms to recover one coefficient, when  $\mathbf{G}$  is obtained from a Gaussian ensemble and the non-zero coefficients are realized from a uniform distribution with  $K_x = 100$  and  $K_d = 100$ .